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# Buckling of Symmetrically Laminated Plates with Compression, Shear, and In-Plane Bending

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**A parametric study of the buckling behavior of infinitely long symmetrically laminated anisotropic plates subjected to combined loadings is presented. The loading conditions considered are pure in-plane bending, transverse tension and compression, and shear. Results obtained using a special purpose analysis that is well suited for parametric studies are presented for clamped and simply supported plates. Moreover, results are presented for some common laminate constructions, and generic buckling design charts are presented for a wide range of useful nondimensional parameters.**

## Introduction

PLATE buckling behavior continues to be a research topic of great practical importance. The emergence of structures made of advanced composite materials has given rise to a substantial number of unsolved orthotropic and anisotropic plate buckling problems. An important type of laminated composite plate that has received, and continues to receive, a great deal of attention by researchers is the symmetrically laminated plate. The interest in symmetrically laminated plates is due to the effort currently under way in the aerospace community to reduce the structural weight of aircraft.

In many practical cases, symmetrically laminated plates are orthotropic. However, in some cases these plates exhibit anisotropy in the form of material-induced coupling between pure bending and twisting deformations. This coupling generally yields buckling modes that are skewed in appearance, as depicted in Fig. 1. The effects of orthotropy and anisotropy on the buckling behavior of compression-loaded plates is well known. For example, recent studies that assess the importance of anisotropy on the buckling behavior of compression-loaded symmetrically laminated plates are presented in Refs. 1 and 2. A survey of anisotropic plate buckling studies that were conducted before 1986 is given in Ref. 3. Two studies that address the generic effects of orthotropy and anisotropy on the buckling behavior of shear-loaded plates and plates subjected to biaxial compression, shear, and in-plane bending loads are presented in Refs. 4 and 5. However, the effects of orthotropy, and particularly anisotropy, on symmetrically laminated plates subjected to in-plane bending loads and various other combined loadings are generally not well known.

Symmetrically laminated plates exhibit a wide range of buckling resistance due to the wide variety of material systems, fiber orientations, and stacking sequences that can be used to construct a laminate. Thus, it is extremely useful for the designer to be able to rapidly compare the performance of various laminated plates through the use of nondimensional parameters and generic buckling curves. Curves of this type can be constructed that span the complete range of plate dimensions, loading combinations, boundary conditions, laminate construction, and material properties. An added benefit of this approach is that the generic curves also furnish the designer with an overall indication of the sensitivity of the structural response to changes in the design parameters. Examples of generic design charts, for buckling and postbuckling of orthotropic plates, that use nondimensional parameters are presented in Refs. 4–6.

Buckling analysis and the behavior of infinitely long plates have two aspects that are of practical importance. First, results for infinitely long plates often provide a lower bound on results for corresponding plates with finite length. Thus, results for infinitely long plates, which are typically easier to obtain than results for finite-length plates, can be useful for preliminary design. Moreover, results for infinitely long plates can be useful for explaining the behavior of corresponding finite-length plates. Second, knowledge of the buckling behavior of infinitely long plates can provide insight into the buckling behavior of more complex structures such as stiffened panels. These two considerations motivated the present study.

The objectives of the present paper are to indicate the effects of plate bending orthotropy and anisotropy on the buckling of infinitely long plates subjected to in-plane bending loads and combined loads and to present generic curves that concisely indicate the buckling behavior of infinitely long plates for a wide range of parameters. The loading conditions considered in this paper are pure in-plane bending, transverse tension and compression, and shear loadings. Results are presented herein for infinitely long plates with the two opposite long edges clamped or simply supported. A substantial number of generic buckling curves that are applicable to a wide range of laminate constructions are presented in the present paper.

## Parametric Study Approach

The buckling behavior of plates is presented in the present study in two different ways. First, the effects of plate orthotropy and anisotropy are presented in an implicit manner for symmetrically laminated angle-ply plates. This family of

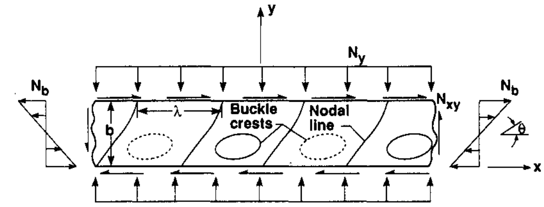


**Fig. 1 Geometry of a buckled plate subjected to pure in-plane bending, transverse compression, and shear loadings.**

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laminates is denoted herein by the symbols $[\pm\theta]_s$, where $\theta$ is the fiber orientation angle shown in Fig. 1 (given in units of degrees). This family of laminates was chosen since varying the laminate parameter $\theta$ encompasses a wide range of plate bending orthotropy and bending anisotropy. The term implicit was used earlier to reflect the fact that the plate orthotropy and anisotropy are implicit functions of laminate fiber orientations and stacking sequence and do not typically vary independently. This implicit way of assessing plate behavior is generally the approach that is the most familiar to aerospace vehicle designers.

A second way to characterize the effects of plate orthotropy and anisotropy uses the nondimensional parameters presented in Refs. 1 and 5. These nondimensional parameters are used in the present paper to show the effects of plate orthotropy and anisotropy on buckling behavior in a somewhat more explicit manner. For example, buckling results are presented in the present paper as a function of two parameters that characterize plate orthotropy and two parameters that characterize plate anisotropy. By varying each parameter in a systematic manner, the effects of orthotropy and anisotropy on buckling can be assessed independently. An important step in this assessment is to establish an understanding of how laminate fiber orientations, stacking sequence, and material properties affect the values of the nondimensional parameters. A detailed discussion of these considerations is presented in Ref. 5.

## Nondimensional Parameters and Analysis Description

Often in preparing generic design charts for buckling of a single flat plate, a special purpose analysis is preferred over a general purpose analysis code, such as a finite element code, due to the cost and effort usually involved in generating a large number of results with a general purpose analysis code. The results presented in the present paper have been obtained using such a special purpose analysis. A brief description of the analysis is presented subsequently.

The buckling analysis used in the present study is based on the classical Rayleigh-Ritz variational method. The special purpose analysis is not new in principle but is derived explicitly in terms of the nondimensional parameters defined in Refs. 1 and 5. The recasting of classical buckling analysis into a nondimensional form was motivated by the need to conduct in-depth parametric studies of buckling behavior that are generic. Furthermore, deriving the analysis in this manner inherently makes the resulting computer code well suited for parametric studies. The nondimensional parameters used are given by

$$\alpha = \frac{b}{\lambda}(D_{11}/D_{22})^{1/4} \tag{1}$$

$$\beta = \frac{D_{12} + 2D_{66}}{\sqrt{D_{11}D_{22}}} \tag{2}$$

$$\gamma = \frac{D_{16}}{(D_{11}^3 D_{22})^{1/4}} \tag{3}$$

$$\delta = \frac{D_{26}}{(D_{11}D_{22}^3)^{1/4}} \tag{4}$$

where $\lambda$ is the half-wavelength of the buckle mode and $b$ is the plate width shown in Fig. 1. The subscripted $D$ terms appearing in the equations are the plate bending stiffnesses of classical laminated plate theory. The parameters $\alpha$ and $\beta$ characterize plate bending orthotropy, and the parameters $\gamma$ and $\delta$ characterize plate bending anisotropy. The parameters defined by Eqs. (2-4) depend only on the plate bending stiffnesses, whereas the parameter $\alpha$ depends on the buckle aspect ratio $\lambda/b$ also. Without loss of generality, and as a matter of convenience, the nondimensional parameter $(D_{11}/D_{22})^{1/4}$ is used in the present study in the place of the parameter $\alpha$ to assess the effects of plate bending orthotropy.

In addition to $(D_{11}/D_{22})^{1/4}$, $\beta$, $\gamma$, and $\delta$, three additional nondimensional quantities are used in the present study to characterize buckling resistance. The quantities are given by

$$K_b = \frac{N_b b^2}{\pi^2\sqrt{D_{11}D_{22}}} \tag{5}$$

$$K_y = \frac{N_y b^2}{\pi^2 D_{22}} \tag{6}$$

$$K_s = \frac{N_{xy} b^2}{\pi^2 (D_{11}D_{22}^3)^{1/4}} \tag{7}$$

The quantities $K_b$, $K_y$, and $K_s$ are referred to herein as the buckling coefficient associated with pure in-plane bending, the transverse compression buckling coefficient, and the shear buckling coefficient, respectively. The stress resultants appearing in the buckling coefficients correspond to critical values associated with the onset of buckling. Each of the loading conditions associated with these buckling coefficients is shown in Fig. 1. The positive-valued stress resultants corresponding to pure in-plane bending, transverse compression, and shear loadings are denoted by $N_b$, $N_y$, and $N_{xy}$, respectively, in the figure.

In the buckling analysis, the infinitely long plates are assumed to have uniform thickness and material properties that do not vary along the plate length and width. In addition, the uniform transverse tension and compression and shear loadings and the boundary conditions do not vary along the plate length. Under these conditions, infinitely long plates have periodic buckling modes that exhibit either inversion symmetry or inversion antisymmetry with respect to a given reference point. However, when pure in-plane bending loads are present, the buckle pattern exhibits a generally asymmetric shape. The buckling mode depicted in Fig. 1 corresponds to this periodic asymmetric shape. In the figure, the buckle pattern is skewed, and the buckle crests are not centered across the plate width.

The mathematical expression used in the variational analysis to describe the generally asymmetric buckle pattern is given by

$$w_N(\xi, \eta) = \sum_{m=1}^{N} (A_m \sin \pi\xi + B_m \cos \pi\xi)\Phi_m(\eta) \tag{8}$$

where $\xi = x/\lambda$ and $\eta = y/b$ are nondimensional coordinates (see Fig. 1), $w_N$ is the out-of-plane displacement field, and $A_m$ and $B_m$ are the unknown displacement amplitudes. In accordance with the Raleigh-Ritz method, the basis functions $\Phi_m(\eta)$ are required to satisfy the kinematic boundary conditions on the plate edges at $\eta = 0$ and 1. For the simply supported plates, the basis functions used in the analysis are given by

$$\Phi_m(\eta) = \sin m\pi\eta \tag{9}$$

for values of $m = 1, 2, 3, \ldots, N$. Similarly, for the clamped plates, the basis functions are given by

$$\Phi_m(\eta) = \cos(m-1)\pi\eta - \cos(m+1)\pi\eta \tag{10}$$

Algebraic equations governing buckling of infinitely long plates are obtained by substituting the series expansion for the buckling mode given by Eq. (8) into the second variation of the total potential energy and then computing the integrals appearing in the second variation in closed form. The resulting equations constitute a generalized eigenvalue problem that depends on the aspect ratio of the buckle pattern $\lambda/b$ (see Fig. 1) and the nondimensional parameters defined by Eqs. (1-4). The in-plane loadings are expressed in terms of a loading parameter that is increased monotonically until buckling occurs. The smallest eigenvalue of the problem corresponds to buckling and is found by specifying a value of $\lambda/b$ and solving the corresponding generalized eigenvalue problem for its smallest eigenvalue. This process is repeated for successive values of $\lambda/b$ until the absolute smallest eigenvalue is found.

Results obtained using the analysis described in the present paper were compared with results for isotropic and specially orthotropic plates published in the technical literature (see Refs. 7-16). The comparisons include results for clamped and simply supported plates and results for axial compression, pure in-plane bending, shear, and transverse compression loadings. The comparisons also include results for combinations of the loadings. Selected results for anisotropic plates were compared with results obtained using the computer code VIPASA (see Ref. 17 for a description of VIPASA). For all these comparisons, results obtained from the analysis used in the present study were found to agree, within a few percent at worst, with the results published in Refs. 7-16 and the results obtained using VIPASA.

## Laminate Construction and Parameter Values

To characterize plate buckling behavior in terms of nondimensional parameters, it is important to understand how laminate fiber orientation, stacking sequence, and material properties affect the values of the nondimensional parameters. In addition, it is important to know the range of values that the nondimensional parameters possess for practical laminate constructions. To determine a representative numerical range of the nondimensional parameters for typical balanced symmetric laminates, results were obtained for $[(\pm\theta)_m]_s$ angle-ply laminates, $[(\pm 45/0/90)_m]_s$ and $[(0/90/\pm 45)_m]_s$ quasi-isotropic laminates, and $[(\pm 45/0_2)_m]_s$ and $[(\pm 45/90_2)_m]_s$ laminates, typically referred to as orthotropic laminates, for integer values of $m$ ranging from 1 to 12 in some cases. The effects of these fiber orientations and stacking sequences on the values of the nondimensional parameters were determined for laminates made of a typical graphite-epoxy material, having a longitudinal modulus $E_1 = 127.8$ GPa ($18.5 \times 10^6$ psi), a transverse modulus $E_2 = 11.0$ GPa ($1.6 \times 10^6$ psi), an in-plane shear modulus $G_{12} = 5.7$ GPa ($0.832 \times 10^6$ psi), major Poisson's ratio $\nu_{12} = 0.35$, and a nominal ply thickness of 0.127 mm (0.005 in.). In addition, the effects of varying the material properties on the values of the nondimensional parameters for $[\pm\theta]_s$ angle-ply laminates were determined for several different material systems. These material systems include high-strength graphite-epoxy material, ultra-high-modulus graphite-epoxy material, S-glass-epoxy material, Kevlar-epoxy material, boron-epoxy material, and boron-aluminum material. A detailed account of these results is presented in Ref. 5.

The results of Ref. 5 briefly described in this section indicate a practical range of the numerical values of the nondimensional parameters. In subsequent sections of this paper, generic buckling results are presented for $0.2 \le \beta \le 3.0$, $0.5 \le (D_{11}/D_{22})^{1/4} \le 3.0$, $0 \le \gamma \le 0.8$, and $0 \le \delta \le 0.8$. A value of 0.8 for $\gamma$ and $\delta$ corresponds to an extremely anisotropic plate. Negative values for $\gamma$ and $\delta$ are possible for some laminates (e.g., $[\mp 45]_s$ laminates) but are not considered in the present paper. Moreover, results are presented in which the nondimensional parameters are varied independently in a systematic manner. In using this approach, the values of the nondimensional parameters are required to obey the condition that the strain energy of the plate be positive valued. This condition corresponds to the thermodynamic restrictions that define the admissible range of plate bending stiffnesses. This condition is enforced in the buckling analysis and accounts for the fact that some of the curves shown on the generic buckling design charts presented herein do not span the entire range of the nondimensional parameters specified herein.

## Results and Discussion

Results are presented in this section for clamped and simply supported plates loaded by pure in-plane bending, combined pure in-plane bending and shear, and combined transverse tension or compression and pure in-plane bending. In loading cases involving shear, a distinction is made between positive and negative shear loadings whenever anisotropy is present. A positive shear loading corresponds to a shear stress acting on the plate edge $y = b$ (see Fig. 1) in the positive $x$ coordinate direction. No distinction between positive and negative pure in-plane bending loads is necessary for anisotropic plates. For each of the loading cases considered, results are first presented for the familiar angle-ply laminates, and then generic results are presented in terms of the nondimensional parameters described herein. The results presented for the angle-ply laminates are calculated using lamina material properties of the typical graphite-epoxy materials previously described herein.

### Plates Subjected to Pure In-Plane Bending Loads

Nondimensional buckling coefficients $K_b$ are presented in Fig. 2 for infinitely long clamped and simply supported $[\pm\theta]_s$ plates loaded in pure in-plane bending, as a function of the fiber orientation angle $\theta$. The four-ply $[\pm\theta]_s$ laminates are limiting cases, but they are useful in the present study since they exhibit the highest degree of anisotropy of the angle-ply laminates. The solid lines shown in Fig. 2 correspond to results for which anisotropy is included in the analysis. The dashed lines in the figure correspond to results for which the $D_{16}$ and $D_{26}$ anisotropic constitutive terms are neglected. The buckling coefficients shown in Fig. 2 were calculated for increments in $\theta$ equal to 1 deg. Similar sets of results for plates loaded by axial compression only and plates loaded by shear only are presented in Ref. 5.

The results presented in Fig. 2 indicate that neglecting plate anisotropy in the analysis always yields nonconservative estimates of the buckling resistance. This trend is due to the fact that neglecting anisotropy overestimates the apparent axial stiffness associated with out-of-plane deformation. The largest buckling coefficients for the orthotropic plates occur at $\theta = 45$ deg, whereas the largest buckling coefficients for the anisotropic plates occur for $\theta$ slightly less than 45 deg. The largest difference between the orthotropic and anisotropic solutions for both clamped and simply supported plates is about 25% of the corresponding orthotropic solution. Comparing these results with the corresponding results presented in Ref. 5 for plates loaded by uniform axial compression indicates that plates subjected to these two loading cases exhibit practically the same trends. This observation is consistent with the fact that the destabilizing forces for both of these loading conditions are compressive and act in the axial direction. Moreover, the results presented in Ref. 5 for the corresponding shear-loaded plates and the results presented herein indicate that anisotropy is more important for shear-loaded plates than for compression-loaded plates or for plates loaded by pure in-plane bending.

Generic buckling results were obtained for clamped and simply supported plates subjected to pure in-plane bending loads for a wide range of values of the nondimensional parameters. The buckling coefficients were plotted as a function of $(D_{11}/D_{22})^{1/4}$ for values of the anisotropic parameters $\gamma$ and $\delta$ ranging from 0 to 0.8. These plots show a family of horizontal straight lines with each line corresponding to specific values of $\gamma$ and $\delta$. These results indicate that the buckling coefficients are independent of the parameter $(D_{11}/D_{22})^{1/4}$ for

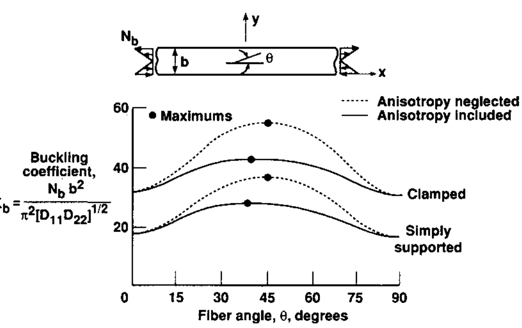


**Fig. 2 Buckling coefficients for $[\pm\theta]_s$ graphite-epoxy laminates loaded by pure in-plane bending.**

the entire range of anisotropy considered. This finding is also exhibited by the corresponding compression-loaded and shear-loaded plates discussed in Ref. 5. This finding is consistent with the observation that plots of buckling coefficient vs plate aspect ratio (length over width), for finite-length plates, attenuate to constant values as the plate aspect ratio becomes large. More specifically, in a finite-length plate, the bending stiffness ratio $(D_{11}/D_{22})^{1/4}$ influences the size and number of buckles that occur along the plate length. The number of buckles that occur along the plate length, in turn, directly affect the value of the buckling coefficient for a given plate (as indicated by the festoon nature of the buckling curves for finite-length plates). For an infinitely long plate, the bending stiffness ratio $(D_{11}/D_{22})^{1/4}$ still influences the size of the buckles, with respect to the basic periodic unit of an infinitely long plate, but the number of buckles that occur along the plate length becomes less significant. This independence of $K_b$ with respect to $(D_{11}/D_{22})^{1/4}$ represents an important simplification in that the buckling coefficients of long plates can be represented by a single orthotropic parameter, namely $\beta$. However, the parameter $(D_{11}/D_{22})^{1/4}$ is important since it does affect the aspect ratio of the buckle pattern and is useful in determining when classical plate theory becomes insufficient and transverse shear deformation must be included in buckling analyses.

Generic buckling coefficients are presented in Fig. 3 for clamped plates loaded by pure in-plane bending as a function of the orthotropic parameter $\beta$ for values of $\gamma = \delta$ ranging from 0 to 0.8. The results shown in this figure are applicable to a large number of laminates. Results for some common laminates made of a typical graphite-epoxy material and an ultra-high-modulus graphite-epoxy material (see Ref. 5) are shown in Fig. 3 and are indicated by symbols. The results

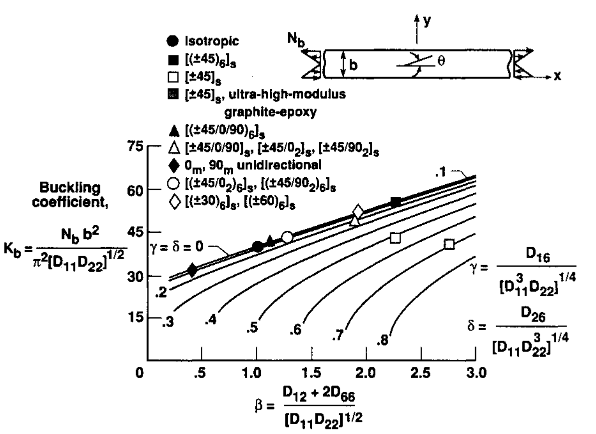


**Fig. 3 Effects of orthotropic parameter $\beta$ and anisotropic parameters $\gamma$ and $\delta$ on buckling coefficients for clamped plates loaded by pure in-plane bending. All laminates shown are made of typical graphite-epoxy material unless otherwise noted.**

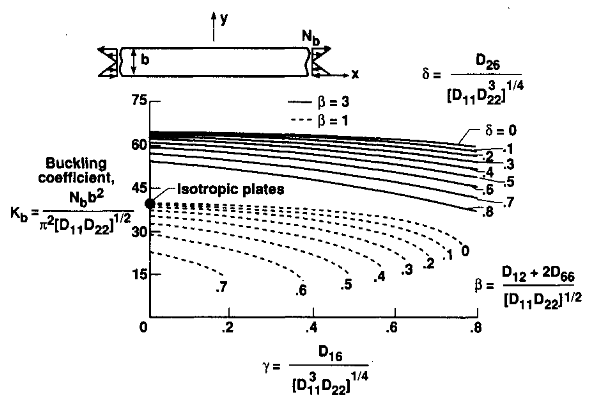


**Fig. 4 Effects of anisotropic parameters $\gamma$ and $\delta$ on buckling coefficients for clamped plates loaded by pure in-plane bending.**

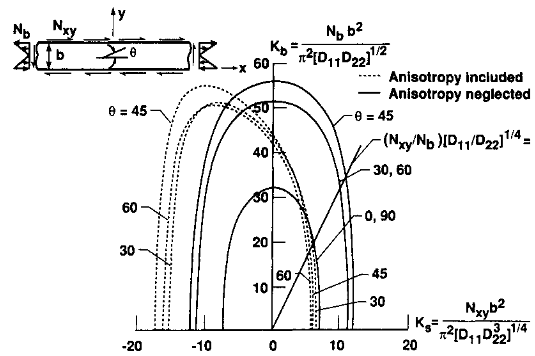


**Fig. 5 Buckling interaction curves for clamped $[\pm\theta]_s$ graphite-epoxy laminates subjected to pure in-plane bending and shear loadings.**

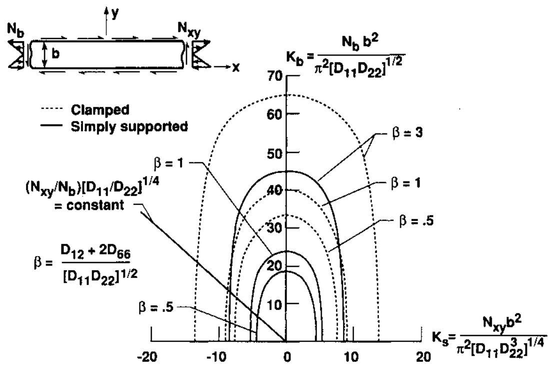


**Fig. 6 Effect of orthotropic parameter $\beta$ on the buckling interaction curves for specially orthotropic laminates ($\gamma = \delta = 0$) subjected to pure in-plane bending and shear.**

presented in Fig. 3 indicate that anisotropy can significantly reduce buckling resistance and that the effect of anisotropy tends to diminish somewhat as $\beta$ increases (i.e., the curves become spaced closer together as $\beta$ increases). The parameter $\beta$ is a measure of a plate's ability to resist twisting deformations and deformations associated with anticlastic curvature effects. The results in Fig. 3 suggest that, as $\beta$ increases, the corresponding increase in stiffness becomes somewhat more dominant than the loss in stiffness due to anisotropy. Similar results were obtained for simply supported plates. These results indicate the same trends and indicate that the effects of varying $\beta$ are slightly more pronounced for the clamped plates (i.e., the linear portions of the curves for the clamped plates generally have a higher slope than the corresponding curves for the simply supported plates). Furthermore, these trends are the same as those presented in Ref. 5 for corresponding axial-compression-loaded plates and for shear-loaded plates. This greater sensitivity of clamped plates to the parameter $\beta$ is due to the increase in a plate's resistance to twisting deformations associated with restraining the plate rotations at the edges; i.e., the twisting resistance becomes more fully developed.

The results in Fig. 3 also give some useful qualitative information for some specific laminates. For example, the results in Fig. 3 show that the buckling coefficients for the $[\pm 45]_s$ plates are larger than those for the unidirectional laminates and approximately the same value as those for the isotropic and $[(\pm 45/0/90)_6]_s$ quasi-isotropic laminates.

Results are presented in Fig. 4 for clamped plates loaded by pure in-plane bending in which the anisotropic parameters $\gamma$ and $\delta$ are varied independently from 0 to 0.8. Two sets of curves are shown in Fig. 4 corresponding to values of the orthotropic parameter $\beta$ equal to 1 and 3. The results presented in this figure indicate that the buckling coefficients are generally more sensitive to the parameter $\delta$ than to $\gamma$. This

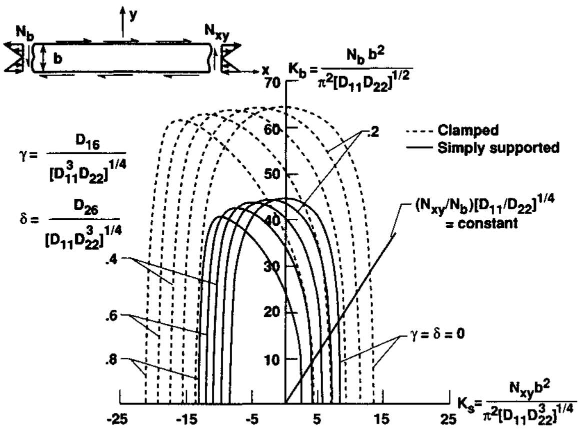


**Fig. 7 Effect of anisotropic parameters $\gamma$ and $\delta$ on the buckling interaction curves for laminates subjected to pure in-plane bending and shear ($\beta = 3$).**

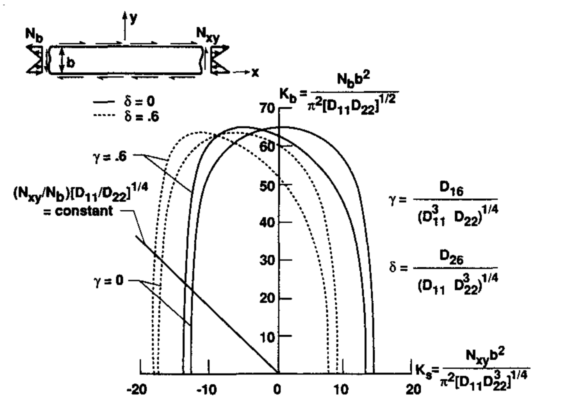


**Fig. 8 Effect of anisotropic parameters $\gamma$ and $\delta$ on the buckling interaction curves for clamped laminates subjected to pure in-plane bending and shear ($\beta = 3$).**

observation can be seen by comparing the differences in the values of the buckling coefficients in which $\gamma$ is held constant and $\delta$ is varied and vice versa. The greater sensitivity of the buckling coefficients to the parameter $\delta$ than to $\gamma$ is due to a boundary effect; i.e., the buckling resistance of an anisotropic plate loaded in uniform axial compression, or in-plane bending, is dependent on the effect of its twisting stiffness on its apparent axial stiffness. It seems reasonable that the long edges of clamped and simply supported plates have a dominant influence on the twisting stiffness and that this effect is associated with the anisotropic stiffness $D_{26}$. The results also show that the effects of anisotropy tend to diminish somewhat as $\beta$ increases, for a wide range of anisotropy. Corresponding results for simply supported plates indicate the same trends and indicate that the effects of varying $\beta$, $\gamma$, and $\delta$ are also slightly more pronounced for the clamped plates. These trends are also the same as those presented in Ref. 5 for corresponding axial-compression-loaded plates and for shear-loaded plates.

### Plates Subjected to Combined Loadings

Results are presented in this section for plates subjected to combined loadings. First, results are presented for plates subjected to combined pure in-plane bending and shear. Next, results are presented for plates subjected to combined transverse tension or compression and pure in-plane bending. In Figs. 5–12, a straight line emanating from the origin of the plot is shown. Points of intersection of the buckling interaction curves with any straight line emanating from the origin constitute constant values of a stiffness-weighted loading ratio. This ratio is obtained from the ratio of the buckling coefficients. These lines are useful for comparing buckling resistance of various plates for a specific applied loading. A typical line and the corresponding value of the stiffness-weighted loading ratio is given in each figure for convenience.

Generic buckling results were obtained for a wide range of the nondimensional parameters for clamped and simply supported plates subjected to each combined load case. Plots were made for several specific combinations of $\beta$, $\gamma$, and $\delta$ in which $(D_{11}/D_{22})^{1/4}$ was varied. The buckling interaction curves corresponding to each combination of $\beta$, $\gamma$, and $\delta$ with different values of $(D_{11}/D_{22})^{1/4}$ are identical. These results indicate that the buckling interaction curves are independent of the parameter $(D_{11}/D_{22})^{1/4}$.

#### *Pure In-Plane Bending and Shear Loading*

Buckling interaction curves are presented in Fig. 5 for infinitely long clamped $[\pm\theta]_s$ plates subjected to pure in-plane bending and shear loadings for several values of fiber orientation angle $\theta$. The dashed lines shown in Fig. 5 correspond to results for which anisotropy is included in the analysis. The solid lines in the figure correspond to results for which the $D_{16}$ and $D_{26}$ anisotropic constitutive terms are neglected. Negative values of $K_s$ correspond to negative shear loadings. Points of intersection of the interaction curves shown in the figure with a straight line emanating from the origin of the plot constitute constant values of $(N_{xy}/N_b)(D_{11}/D_{22})^{1/4}$.

Generic buckling interaction curves are presented in Figs. 6–8 for clamped and simply supported plates. Results are presented in Fig. 6 for several values of the orthotropic parameter $\beta$ and for values of $\gamma = \delta = 0$. Thus, the results characterize the effects of orthotropy on the buckling resistance. The results presented in this figure are applicable to a large number

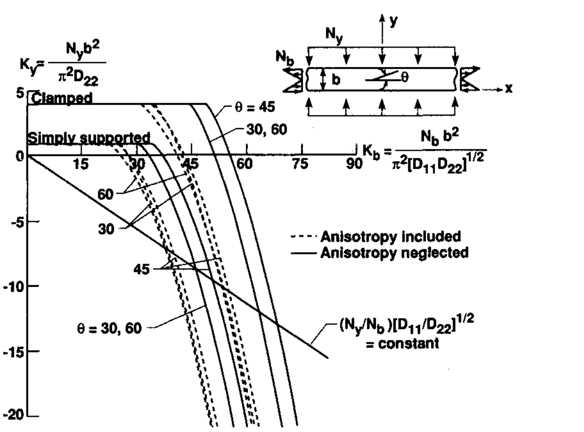


**Fig. 9 Buckling interaction curves for $[\pm\theta]_s$ graphite-epoxy laminates subjected to transverse tension or compression and pure in-plane bending.**

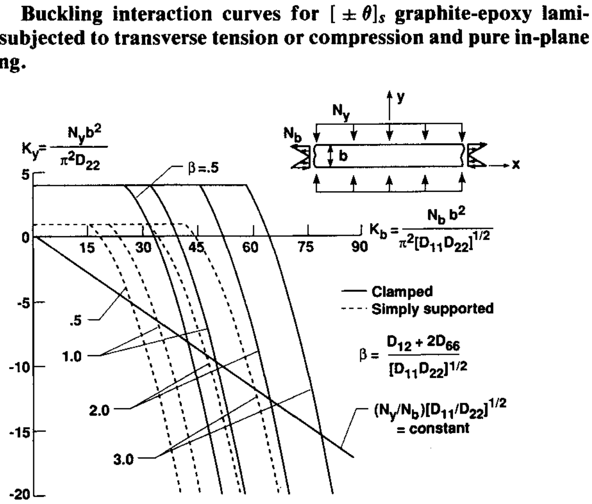


**Fig. 10 Effect of orthotropic parameter $\beta$ on the buckling interaction curves for specially orthotropic laminates ($\gamma = \delta = 0$) subjected to transverse tension or compression and pure in-plane bending.**

of specially orthotropic plates and to plates that have anisotropic parameters whose values are small compared with unity. For example, results for isotropic plates are given by the curve for $\beta = 1$. Results for $[(\pm 45)_{12}]_s$ plates made from a typical graphite-epoxy material would be given approximately by a curve corresponding to $\beta = 2.3$ that can be obtained by interpolation. Results showing the generic effects of anisotropy are presented in Figs. 7 and 8 for clamped and simply supported plates with $\beta = 3$. Specifically, results are presented in Fig. 7 in which the anisotropic parameters $\gamma$ and $\delta$ have the same value that ranges from 0 to 0.8. Results are presented in Fig. 8 for clamped plates in which $\gamma$ and $\delta$ are varied independently. Curves are shown in this figure for values of $\gamma$ and $\delta$ equal to combinations of 0 and 0.6.

The results presented in Figs. 5 and 6 show that the effects of orthotropy manifest themselves as a self-similar enlargement or a contraction of the buckling interaction curves that is symmetric about the $K_b$ axis. In addition, the results presented in Fig. 6 indicate that substantial gains in buckling resistance can be obtained by tailoring a laminate construction to increase the parameter $\beta$; i.e., increase the plate twisting stiffness and resistance to anticlastic deformations. The results presented in Figs. 5, 7, and 8 indicate that the effects of anisotropy manifest themselves as a horizontal shift and skewing of the buckling interaction curves in the $K_b$-$K_s$ plane. The horizontal shift is similar to the trend exhibited by the plates subjected to axial compression and shear loading described in Ref. 5. However, the curves shown in Ref. 5 did not exhibit skewing. For plates loaded in positive shear, neglecting plate anisotropy may greatly overestimate the buckling resistance. For example, neglecting the anisotropy in the analysis of a $[\pm 45]_s$ clamped plate with $(N_{xy}/N_b)(D_{11}/D_{22})^{1/4} = 0.3$ (see Fig. 5) results in a buckling coefficient that is nearly twice the actual anisotropic buckling coefficient. The overestimation of the buckling resistance generally decreases as the amount of shear in the loading combination diminishes. For both clamped and simply supported plates that are loaded in positive shear and pure in-plane bending (see Fig. 7), and with $\gamma = \delta = 0.8$, neglecting anisotropy in a buckling analysis yields buckling coefficients that range from approximately 40 to 70% of the corresponding orthotropic buckling coefficient. The smaller percentage corresponds to plates loaded by pure in-plane bending only, and the larger percentage corresponds to plates loaded by positive shear only.

The results presented in Figs. 5, 7, and 8 indicate a different trend for plates loaded in negative shear than for plates loaded in positive shear. These results suggest that the buckling resistance of a plate loaded by pure in-plane bending can be increased by applying a negative shear loading. For example, the results for clamped plates with $\theta = 45$ deg shown in Fig. 5, and corresponding results for simply supported plates, exhibit an increase of approximately 25% of the $K_b$ values corresponding to pure in-plane bending load only. Similarly, Fig. 7 indicates that the addition of negative shear can also increase the buckling coefficients of extremely anisotropic clamped and simply supported plates ($\gamma = \delta = 0.8$) by approximately 70%. The results presented in Fig. 7 also indicate that the orthotropic plates ($\gamma = \delta = 0$) are the most buckling resistant of the plates loaded by positive shear forces. The increase in buckling resistance associated with the addition of negative shear loading arises from the fact that the equivalent diagonal forces associated with the shear loading act to oppose the compressive force due to the in-plane bending load.

The results presented in Fig. 8 for clamped plates indicate the same trend shown in Fig. 4 for the plates loaded by pure in-plane bending; i.e., the buckling coefficients are generally more sensitive to the anisotropic parameter $\delta$ than to $\gamma$. Corresponding results for simply supported plates also exhibited the same trend.

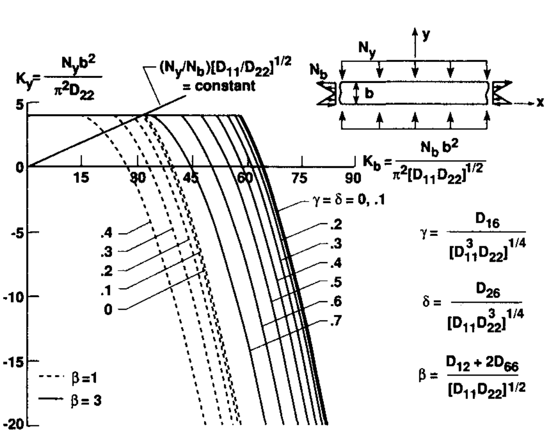


**Fig. 11 Effect of anisotropic parameters $\gamma$ and $\delta$ on the buckling interaction curves for clamped laminates subjected to transverse tension or compression and pure in-plane bending.**

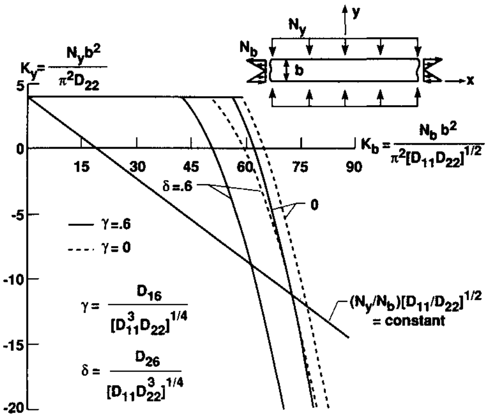


**Fig. 12 Effect of anisotropic parameters $\gamma$ and $\delta$ on the buckling interaction curves for clamped laminates subjected to transverse tension or compression and pure in-plane bending.**

### *Transverse Tension or Compression and Pure In-Plane Bending*

Buckling interaction curves are presented in Fig. 9 for infinitely long clamped and simply supported $[\pm \theta]_s$ plates subjected to combined transverse tension or compression and pure in-plane bending loads for several values of fiber orientation angle $\theta$. The dashed lines shown in Fig. 9 correspond to results for which anisotropy is included in the analysis, and the solid lines correspond to results for which it is neglected. Points of intersection of the interaction curves shown in the figure with a straight line emanating from the origin of the plot constitute constant values of $(N_y/N_b)(D_{11}/D_{22})^{1/2}$. The horizontal straight-line portions of the buckling interaction curves indicate values of the buckling coefficients at which a plate buckles into a wide-column mode.

Generic buckling interaction curves are presented in Figs. 10–12 for clamped and simply supported plates. Generic results for specially orthotropic plates ($\gamma = \delta = 0$) are presented in Fig. 10 for several values of the orthotropic parameter $\beta$. Results showing the effects of varying the anisotropic parameters are presented in Fig. 11 for clamped plates with values of $\gamma = \delta$ ranging from 0 to 0.7. The solid and dashed lines shown in the figure correspond to values of $\beta = 3$ and 1, respectively. Similarly, results are presented in Fig. 12 for clamped plates with $\beta = 3$ in which the anisotropic parameters $\gamma$ and $\delta$ are varied independently; i.e., combinations of 0 and 0.6.

The results presented in Figs. 9 and 10 indicate that the effects of orthotropy on plates subjected to these loading conditions are generally manifested as a shift of self-similar buckling interaction curves along the $K_b$ axis. The results shown in Fig. 10 indicate the same trend that is exhibited by the corresponding plates loaded by combined pure in-plane bending and shear; i.e., substantial gains in buckling resistance can be obtained with increases in the orthotropic parameter $\beta$ (increases in twisting stiffness and resistance to anticlastic deformations). The results presented in Figs. 9, 11, and 12 indicate that the effects of anisotropy for plates subjected to these loading conditions also generally manifest themselves as a shift of the interaction curves along the $K_b$ axis. Moreover, the results indicate that neglecting plate anisotropy in an analysis always yields buckling coefficients larger than those corresponding to the anisotropic solution, but not to the degree exhibited by the corresponding plates loaded by pure in-plane bending and shear. This trend is consistent with the sensitivity of shear buckling resistance to anisotropy previously indicated herein. For the $[\pm 45]_s$ laminates, the overestimation in the buckling resistance associated with neglecting anisotropy typically ranges from approximately 20 to 30% of the corresponding orthotropic solution (see Fig. 9). The larger percentage corresponds to differences obtained from solutions that predict the onset of a wide-column buckling mode. Similarly, the generic results presented in Fig. 11 for clamped plates show that the degree of overestimation is slightly dependent on the $K_y$-to-$K_b$ ratio and ranges from approximately 30 to 40% of the corresponding orthotropic results at most for the range of parameters indicated in the figure. The higher percentage also corresponds to buckling coefficients associated with the onset of a wide-column buckling mode. Results for corresponding simply supported plates indicate the same trends. The generic results presented in Fig. 11 for clamped plates, and corresponding results for simply supported plates, also show that the effects of anisotropy are more pronounced for plates with $\beta = 1$ than for plates with $\beta = 3$. The results presented in Fig. 12 indicate the plates loaded by combined transverse tension or compression and pure in-plane bending are also generally more sensitive to the parameter $\delta$ than to $\gamma$. Corresponding results for simply supported plates also indicate the same trends.

## Concluding Remarks

A parametric study of the buckling behavior of infinitely long symmetrically laminated anisotropic plates subjected to combined loadings has been presented. Loading conditions consisting of pure in-plane bending, transverse tension and compression, and shear were investigated for clamped and simply supported plates. Results were obtained using a special purpose analysis that was derived in terms of useful nondimensional parameters. Buckling results have been presented for some common laminate constructions, and generic buckling design charts have been presented for a wide range of nondimensional parameters.

The present paper shows that nondimensional parameters can be very useful for concisely presenting results for a wide range of loading conditions, boundary conditions, and laminate constructions. Results show that the effects of anisotropy are much more pronounced in shear-loaded plates than in compression-loaded plates and plates loaded by pure in-plane bending. In addition, the effects of anisotropy on plates subjected to combined loadings are shown to be generally manifested as shifts of the buckling interaction curves. These results indicate that the buckling resistance of long highly anisotropic plates loaded by pure in-plane bending can be increased significantly by applying a shear loading with a specific orientation. However, the results also indicate that the buckling resistance of a plate can be significantly overestimated in some cases by neglecting the anisotropy in a buckling analysis. In addition, it is shown that including anisotropy in a buckling analysis causes larger reductions in the buckling resistance for plates subjected to combined pure in-plane bending and shear than for plates subjected to combined pure in-plane bending and transverse tension or compression.

A substantial number of generic buckling results are presented in the present paper for a wide range of values of the nondimensional parameters. In all of the cases considered, it is found that the buckling coefficients of infinitely long plates are independent of the bending stiffness ratio $(D_{11}/D_{22})^{1/4}$. It is shown that large increases in buckling resistance can be obtained by tailoring the laminate construction to increase the parameter $\beta = (D_{12} + 2D_{66})/(D_{11}/D_{22})^{1/2}$ and that the importance of anisotropy generally diminishes as $\beta$ increases. Results are also presented that show the buckling coefficients to be generally more sensitive to the anisotropic parameter $\delta = D_{26}/(D_{11}D_{22}^3)^{1/4}$ than to the anisotropic parameter $\gamma = D_{16}/(D_{11}^3 D_{22})^{1/4}$ for the entire range of loadings and boundary conditions considered.

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